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ASYMPTOTIC SOLUTIONS OF EQUATIONS OF CLASSICAL MECHANICS*

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Motions of natural mechanical systems that approach their equilibrium positions with unlimited increase of time are considered.

The statement of this problem and indication of existence of asymptotic motions go apparently back to the paper by Kneser /1/, in which the asymptotic trajectories of conservative systems were investigated in the neighborhood of nondegenerate equilibrium positions at which potential energy has a local maximum. Kneser's results were extended in /2/ to the case of degenerate equilibria of nonautonomous mechanical systems. Asymptotic solutions in the general nondegenerate case (when the potential energy Hessian at the equilibrium position is nonzero) were investigated by Bohl /3/. The general theorem stated in Sect.2 below shows the existence of asymptotic motions in degenerate cases, when the force function expansion in the equilibrium position neighborhood begins with odd order terms. Since the equations of motion of natural systems are reversible, the respective equilibrium states are unstable.

1. Equations of motion. Let us consider a natural mechanical system with n degrees of freedom. Let $x \in R^n$ be its generalized coordinates, $T = \langle K(x)x^\cdot, x^\cdot\rangle/2$ the kinetic energy ($\langle,\rangle$ is the scalar product in R^n), and $F(x)$ the generalized forces acting on the system. The equations of motion in Lagrangian form are

$$\frac{d}{dt}\frac{\partial T}{\partial x^\cdot} - \frac{\partial T}{\partial x} = F(x)$$

Since $\det K \neq 0$, these equations can be solved for accelerations

$$x^{\cdot\cdot} = \langle \Gamma(x)x^\cdot, x^\cdot\rangle + f(x), \quad f(x) = K^{-1}(x)F(x)$$

where $\langle \Gamma x^\cdot, x^\cdot\rangle$ is a set of n quadratic forms in velocities (the coefficients of these forms are the Kronecker deltas of the Riemannian metric $\langle K(x)\,dx, dx\rangle$).

The singular points of vector field $f(x)$ are the only positions of equilibrium. We can assume without loss of generality that point $x = 0$ $(f(0) = 0)$ is one of equilibrium, and also that elements of matrix $K(x)$ and vector function $F(x)$ are analytic in some neighborhood of point $0 \in R^n$ (functions $\Gamma(x)$ and $f(x)$ are obviously also analytic in that neighborhood).

We expand vector function $f(x)$ in a convergent power series $f(x) = f_m(x) + f_{m+1}(x) + \ldots$ where $f_p(x)$ is a homogeneous vector function of power p: $f_p(\lambda x) = \lambda^p f_p(x)$. We assumes that $m \geqslant 2$ (the equilibrium position $x = 0$ is degenerate).

2. The theorem on asymptotic motions. We begin by investigating asymptotic solutions of the system of equations $x^{\cdot\cdot} = f_m(x)$ $(m \geqslant 2)$ which we shall call simplified.

Lemma. If for some $e \in R^n(|e| = 1)$ we have $f_m(e) = \varkappa e, \varkappa > 0,$ the simplified equation has the asymptotic solution

$$x(t) = \frac{a}{t^{2/(m-1)}}, \quad a \in R^n, \quad a = |a|e$$

The lemma implies that $f_m(x)$ is a central force repelling along the ray defined by vector e, a condition that is assumed satisfied in what follows. When force $f_m(x)$ is potential and its force function has no local maximum in the equilibrium position, then $f_m(e) = \varkappa e$ for some $e \in R^n$ and $\varkappa > 0$.

Proof. If $x = a/t^{2/(m-1)}$, then

$$x^{\cdot\cdot} = \frac{2(m+1)a}{(m-1)^2\, t^{2m/(m-1)}}$$

On the other hand

$$f_m\left(\frac{|a|e}{t^{2/(m-1)}}\right) = \frac{|a|^m}{t^{2m/(m-1)}} f_m(e) = \frac{|a|^m \varkappa e}{t^{2m/(m-1)}}$$

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Consequently $|a|^{m-1}\varkappa = 2(m+1)/(m-1)^2$ and, since $\varkappa > 0$, vector $a \in R^n$ exists. The asymptotic solution of the complete system

$$x^{\cdot\cdot} = \langle \Gamma(x) x^{\cdot}, x^{\cdot} \rangle + f_m(x) + f_{m+1}(x) + \dots \tag{2.1}$$
$$\Gamma(x) = \Gamma_0(x) + \Gamma_1(x) + \dots, \quad \Gamma_0(x) = \Gamma_0(0) = \text{const}$$

is sought in the form

$$x(t) = \frac{a_1}{t^\mu} + \dots + \frac{a_k}{t^{k\mu}} + \dots; \quad a_k \in R^n, \quad a_1 = a, \quad \mu = \frac{2}{m-1} \tag{2.2}$$

Below, an important part is played by the constant matrix of dimension $n \times n$

$$A = t^2 \frac{\partial f_m}{\partial x}\bigg|_{x=a/t^\mu}$$

Theorem 1. If among the eigenvalues of matrix A there are no numbers

$$\frac{4(m+3)}{(m-1)^2}, \quad \frac{6(m+5)}{(m-1)^2}, \dots, \frac{2k(2k+m-1)}{(m-1)^2}, \dots \tag{2.3}$$

then there exists a unique solution of Eq.(2.1) that can be represented in the form of series (2.2) convergent for fairly large $|t|$.

Corollary 1. If conditions of the above lemma and Theorem 1 are satisfied, the equations of motion have asymptotic solutions in the form of function $x(t)$, which approach zero as $t \to \pm\infty$.

Corollary 2. On these assumptions the equilibrium position $x = 0$ is unstable.

Proof of Theorem 1. We carry out in the equation of motion (2.1) the substitution of time $t \to \tau$ and of the independent variable $x \to y$ using formulas $\tau = \varepsilon t$ and $x = \varepsilon^\mu y$ ($\mu = 2/(m-1)$). The equation then assumes the form

$$y'' - f_m(y) = \varepsilon^\mu [\langle \Gamma(\varepsilon^\mu y) y', y' \rangle + f_{m+1}(y) + \varepsilon^\mu f_{m+2}(y) + \dots]$$

where the prime denotes a derivative with respect to the new "time" τ and the square brackets contain an analytic vector function of y, y' and ε^μ.

Let us set $\varepsilon^{\mu} = \delta$ and consider the equation $F(y'(\tau), \delta) = 0$, where

$$F = y' - \int \{f_m(\textstyle\int y' d\tau) + \delta[\dots]\}\, d\tau$$

and $\int\{\cdot\}\, d\tau$ is the linear operator of formal integration of power series

$$\int \sum_{\lambda_n > 1} \frac{x_n}{\tau^{\lambda_n}}\, d\tau = \sum_{\lambda_n > 1} \frac{x_n}{(1-\lambda_n)\tau^{\lambda_n - 1}}$$

We introduce space B_r of functions $x(\tau)$ that can be represented in the form of series

$$\sum_{k=1}^{\infty} \frac{x_k}{\tau^{k\mu+1}}, \quad x_k \in R^n$$

convergent for $|\tau| > r$, $\tau \in C$ and continuous for $|\tau| \geqslant r$. That space of norm

$$\|x(\tau)\| = \max_{|\tau|=r} |x(\tau)|$$

is a Banach space. For small δ, $\|x(\tau) - y_0'(\tau)\|$ ($y_0 = a/\tau^\mu$) and fairly large $|\tau|$ function $F(x(\tau), \delta)$ (as a function of $\tau \in C$)) belongs to B_r.

The following statements are valid.

1°. Function $F(x(\tau), \delta)$ is the continuous mapping of $U \times (-\varkappa, \varkappa) \to B_r$, where $\varkappa > 0$ is small and U is some fairly small neighborhood of point $x_0 = y_0' = -\mu a/\tau^{\mu+1}$ in the B_r space.

2°. The solution of equation $F(x_0, 0) = 0$ is $x_0 = y_0'$.

3°. Mapping F has in $U \times (-\varkappa, \varkappa)$ the derivative $F_x'(x, \delta)$ which is continuous (at least) at point $(x, \delta) = (y_0', 0)$.

4°. Setting $x = y_0'(\tau) + z(\tau)$, $\delta = 0$ we obtain

$$F_x'(y_0', 0) = z - \int \left\{ \frac{\partial f_{n1}}{\partial y}\bigg|_{y_0(\tau)} \int z\, d\tau \right\} d\tau = z - \int \frac{A}{\tau^2} \left(\int z\, d\tau \right) d\tau$$

Let us show that this linear operator $D: B_r \to B_r$ has a limited inverse. Indeed, let

$$u(\tau) = \sum_{k=1} \frac{u_k}{\tau^{k\mu+1}} \in B_r$$

then the equation

$$z - \int \frac{A}{\tau^2}\left(\int z\,d\tau\right)d\tau = u$$

has the solution

$$z(\tau) = \sum_{k=1}^{\infty} \frac{z_k}{\tau^{k\mu+1}}, \quad z_k = C_k u_k, \quad C_k = \left\| E - \frac{(m-1)^2}{2k(2k+m-1)} A \right\|^{-1}$$

Since $\det C_k \neq 0$ for all k and $C_k = E + O(1/k^2)$, operator D^{-1} exists and is bounded.

The properties $1^\circ - 4^\circ$ of mapping $F(y', \delta)$ enable us to apply the theorem on implicit functions /4/ which states that for small δ equation $F(y'(\tau), \delta) = 0$ has the unique solution $y'(\tau, \delta)$ not much different from function $y_0'(\tau)$. Integrating the power series of function $y'(\tau)$ and reverting to old variables x, t, we obtain the asymptotic solution $x(t)$ of the equations of motion which can be represented by the converging series (2.2).

3. Instability of equilibrium in a potential field. Let us assume that the generalized force $F(x)$ is potential and $V(x)$ is its force function. Let $V(x) = V_{m+1}(x) + V_{m+2}(x) + \ldots (m \geqslant 2)$, where V_p is the homogeneous form of variables $x = (x_1, \ldots, x_n)$ of power $p \in N$. In normal coordinates $x \in R^n$ matrix $K(x)$ which defines the system kinetic energy is of the form $E + \Lambda(x)$, where E is the unit matrix and $\Lambda(0) = 0$. Since $f(x) = K^{-1}(x) F(x)$, hence $f_m(x) = \partial V_{m+1}/\partial x$.

Let $\max_{x \in S} V_{m+1} > 0$, where S is an $(n-1)$-dimensional unit sphere $\{\Sigma x_i^2 = 1\} \subset R^n$, and this maximum is attained on some vector e, then $f_m(e) = \varkappa e$ and $\varkappa > 0$. We can assume without loss of generality that $e = (1, 0, \ldots, 0)$. The form V_{m+1} can be represented as follows:

$$V_{m+1} = wx_1^{m+1} + \sum_{i=2}^{n} v_i(x_1)x_i + \frac{1}{2}\sum_{i,j=1}^{n} v_{ij}(x)x_ix_j, \quad w = \text{const}$$

where v_i and v_{ij} are some homogeneous functions of power m and $m-1$, respectively. Since $\partial V_{m+1}/\partial x_i = 0$ $(i \geqslant 2)$ when $x_i = 0$ $(i \geqslant 2)$, hence $v_i = 0$.
Moreover, for small $x_2, \ldots, x_n$ and if $x_1 = (1 - x_2^2 - \ldots - x_n^2)^{1/2}$ function

$$\sum_{i,j=2}^{n} v_{ij}(x_1, \ldots, x_n)x_ix_j \leqslant 0$$

The simplified equation

$$x^{\cdot\cdot} = f_m(x) = \partial V_{m+1}/\partial x$$

has the solution

$$x_1 = \frac{a}{t^\mu}, \quad a^{m-1} = \frac{2}{w(m-1)^2}; \quad x_i = 0, \quad i \geqslant 2$$

and in this case matrix $A = t^2 \partial f_m/\partial x\,|_{x(t)}$ is of the form

$$\left\| \begin{matrix} \frac{2m(m+1)}{(m-1)^2} & 0 & \ldots & 0 \\ 0 & v_{22}^* & \ldots & v_{2n}^* \\ \vdots & \ldots & \ldots & \ldots \\ 0 & v_{n2}^* & \ldots & v_{nn}^* \end{matrix} \right\|, \quad v_{ij}^* = v_{ji}^* = t^2 v_{ij}(a/t^\mu, 0, \ldots, 0) = \text{const}$$

The eigenvalues $\lambda_2, \ldots, \lambda_n$ of matrix $\|v_{ij}^*\|$, as well as $\lambda_1 = 2m(m+1)/(m-1)^2$ are eigenvalues of matrix A.

Let us ascertain whether numbers $\lambda_1, \ldots, \lambda_n$ belong to sequence (2.3). If for some $k \in N$ the equality $k(2k + m - 1) = m(m+1)$ is satisfied, then $2k^2 + (m-1)k - m(m+1) = 0$, whence $k_1 = -m < 0$, $k_2 = (m+1)/2$. Thus, when expansion of the force function ΣV_i $(i \geqslant 3)$ begins with a nonzero form of odd power, the sequence of numbers (2.3) does not contain λ_1.

Moreover, since the form $\Sigma v_{ij}(1, 0, \ldots, 0)x_ix_j \leqslant 0$, hence $\Sigma v_{ij}^* x_ix_j \leqslant 0$ because v_{ij}^* is proportional to $v_{ij}(1, 0, \ldots, 0)$ with the positive coefficient a^{m-1}. Consequently all $\lambda_2, \ldots, \lambda_n \leqslant 0$.

Thus on the basis of Theorem 1 we have proved the following theorem.

Theorem 2. If the expansion of the force function in a Maclaurin series begins with terms of odd power, there exists an asymptotic motion, and in particular, the equilibrium position $x = 0$ is unstable.

4. Some generalizations. The statements about asymptotic motions proved above are also valid in the more general case, when Chaplygin's nonholonomic mechanical systems are considered instead of holonomic ones.

Indeed, the equations of motion of Chaplygin's system in "canonical" variables $(p, q) \in R^{2n}$ are of the form /5/

$$q^{\cdot} = \frac{\partial T}{\partial p} = K(q)\, p, \quad p^{\cdot} = -\frac{\partial T}{\partial q} + F(q) + \langle B(q)\, p,\ p\rangle \qquad (4.1)$$
$$T = \frac{1}{2}\langle K(q)\, p,\ p\rangle$$

where T is the kinetic energy, $F(q)$ are generalized forces acting at points of the mechanical system and $\langle B(q)\, p, p\rangle$ is some set of n forms quadratic form in momenta $p \in R^n$. These equations are obviously inversible (together with the solution $q(t), p((t)$ and have $q(-t), -p(-t))$ as their solution).

From the system of Eqs.(4.1) we can obtain a second order equation in q

$$q^{\cdot\cdot} = \langle \Gamma(q)\, q^{\cdot},\ q^{\cdot}\rangle + f(q)$$

where $\langle \Gamma q^{\cdot}, q^{\cdot}\rangle$ is a set n of quadratic forms in velocities and $f(q) = K(q)\, F(q)$. By a suitable canonical transformation we can achieve that in the new coordinates matrix $K(q) = E + \Lambda(q),\ \Lambda(0) = 0$.

It remains to point out that the structure of coefficients $\Gamma(q)$ was not used in the proof of Theorem 1.

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